STABLE REARRANGEMENTS OF CONDITIONALLY CONVERGENT SERIES

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ABSTRACT. Analysts have considered the problem of infinite series rearrangement at least since Riemann gave the proof of his series theorem in the 19th century. And in more modern times, series rearrangements have been given a thorough algebraic treatment. Levy established a duality between series $(\alpha_n) \in \mathbf{CC}$ and rearrangements $(\sigma_n) \in S_{\mathbb{N}}$, involuted by the relation "σ fixes α" iff "α is fixed by σ" iff $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} = \sum_{n=1}^{\infty} \alpha_n$. This has been called Levi's duality[2], and significant progress was made in the 1970s to 1980s, especially in the study of fixors of the largest families of series[1]. On the other hand, fixors of small families have not been described with much detail. In this paper, it will be demonstrated that real sequences can be modelled as paths through conservative vector fields (Polya vector fields), and series as the corresponding line (contour) integrals. In this view, series and rearrangements arise from the same fundamental objects, vector fields and paths. One layer deeper, this model is supersymmetric, as it sees a vector field and a path as two manifestations of a single object: a grid-path. A pragmatic approach to finding the fixors of a series will be laid out. Multiple phase transitions in the geometry of this model will be demonstrated, the most crucial being the transition from harmonic fields to sub-harmonic fields.

1. Prologue: Rearranging Series

To illustrate the problem that arises from series rearrangement, consider the token example of the effect of rearrangement on a series. Take the alternating harmonic series $\varrho_n = \frac{(-1)^{n-1}}{n}$,

$$\sum_{k=1}^{\infty} \varrho_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log(2)$$

Date: April 2024.

Thank you to Professor Jon McCammond for your time, for the initial idea of grid representations, continued ideas and help. Thank you to my family for your continued support. Thank you, Lesley.

and the rearrangement σ sums two negative terms after each positive,

$$\begin{aligned}\sum_{k=1}^{n} \varrho_{\sigma_k} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) \\ &= \frac{1}{2}\log(2).\end{aligned}$$

Tricks such as these, which involve some shuffle of a series into positive and negative chunks, are often used to depict that commutativity may fail when infinitely many terms are rearranged. In the context of shuffles, these are actually quite extreme, as they deviate from the given $+, -, +, -, \ldots$ shuffle linearly as $n \to \infty$.

Even so, what if a series which shrinks faster is rearranged in this extreme way? Consider a subharmonic series,

$$\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{kf(k)},$$

where $f(n) \to \infty$. In this case, as will be shown later, the previous rearrangement will not be deviant enough to affect the series.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kf(k)} &= \sum_{j=1}^{\infty} \frac{(-1)^{\sigma_k - 1}}{\sigma_k f(\sigma_k)} \\ &= \frac{1}{f(1)} - \frac{1}{2f(2)} - \frac{1}{4f(4)} + \frac{1}{3f(3)} - \cdots .\end{aligned}$$

For example, let $f(n) = \log(n + 2)$, so that each side converges to about 0.67539049, though the right hand side will converge incredibly slowly, only at about 0.659 after 3 billion terms. We predict that it will attain the third decimal after around 10^{500} terms, but it will converge nonetheless. In general, there is a battle between the growth of the absolute reciprocal sequence $(\frac{1}{|\alpha_n|})$ and the deviation of a shuffle. Intuitively, these series are unaffected by linear shuffles because their reciprocals are superlinear. In fact, many more asymptotic claims can be made about all conditionally convergent series with the aid of the geometric model laid out in the next section.

2. Series and Rearrangements

Consider a conditionally convergent sequence $\alpha = (\alpha_n)$. For a given n, one can partition $\{1, ..., n\}$ into the positive and negative domains $M_n^{\pm} \subset \{1, ..., n\}$ of α,

$$M_n^{\pm} = \{k \leq n \mid \pm\alpha_n > 0\}.$$

$$\begin{aligned}\sum_{j=1}^{n} \alpha_j &= \sum_{j \in M_n^+} \alpha_j + \sum_{k \in M_n^-} \alpha_k \\ &= \sum_{j=1}^{|M_n^+|} \alpha_{m_j^+} - \sum_{k=1}^{|M_n^-|} \alpha_{m_k^-},\end{aligned}$$

where $m_j^{\pm}$ is the unique increasing enumeration of $M_n^{\pm}$, resp. To simplify this correspondence, define

$$\alpha_n^{\pm} = \alpha_{m_n^{\pm}} \text{ and } \mu(n) = (\mu_1(n), \mu_2(n)) = (|M_n^+|, |M_n^-|),$$

yielding

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{\mu_1(n)} \alpha_j^+ - \sum_{k=1}^{\mu_2(n)} \alpha_k^-.$$

Label these signed partial sums $G_1(n)$, $G_2(n)$, respectively and call the resultant formula the standard form of α :

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i &= G_1(\mu_1(n)) - G_2(\mu_2(n)) \\ &= G_1\mu_1(n) - G_2\mu_2(n)\end{aligned}$$

We will use the shorthand $gf(n) = g(f(n))$ for composition of these $\mathbb{N}$ to $\mathbb{N}$ functions. If the need arises to multiply such functions, either $f \cdot g(n)$ or $f(n)g(n)$ will be used. Visually, one can interpret G_1 and G_2 as axes, and μ as an increasing path along the vertices of the lattice $G_1(\mathbb{N}) \times G_2(\mathbb{N})$ embedded in $[0, \infty)^2$.

2.1. **Rearrangements.** Due to our subdivision of $\mathbb{N}$ into $M^{\pm}$ in our treatment of series, it will be beneficial to decompose rearrangements along that same partition.

Proposition 2.1. *Let E, F be two infinite sets partitioning $\mathbb{N}$. Every rearrangement of $\mathbb{N}$ can be uniquely decomposed as a rearrangement of E, a rearrangement of F, and a shuffle between E and F.*

Proof. Let $\sigma \in S_{\mathbb{N}}$. Consider the mapping $\sigma \mapsto (\psi_1, \psi_2, \zeta)$, where $\psi_1 \in S_E$ is its $E-$rearrangement, $\psi_2 \in S_F$ its F-rearrangement, and $\zeta \in \text{Shuf}(E, F)$ its shuffle. This is trivially surjective. For injectivity, let $\sigma, \sigma' \in S_{\mathbb{N}}$. Notice that any shuffle between E and F cannot affect their respective orderings. (a stack of spades will not have its order affected when shuffled once with a stack of hearts) Therefore, their E-rearrangements and F-rearrangements must coincide. Of course, this then determines their shuffles to coincide, as no two distinct shuffles of identical stacks could possibly correspond. QED

3. Modeling Series as Contour Integrals

Let D be the non-negative ray, $D = [0, \infty)$ and $\mathbf{A}_0$ the set of homeomorphisms from D to itself, $\mathbf{A}_0 = \text{Homeo}(D)$. Take $u \in A$ to be the

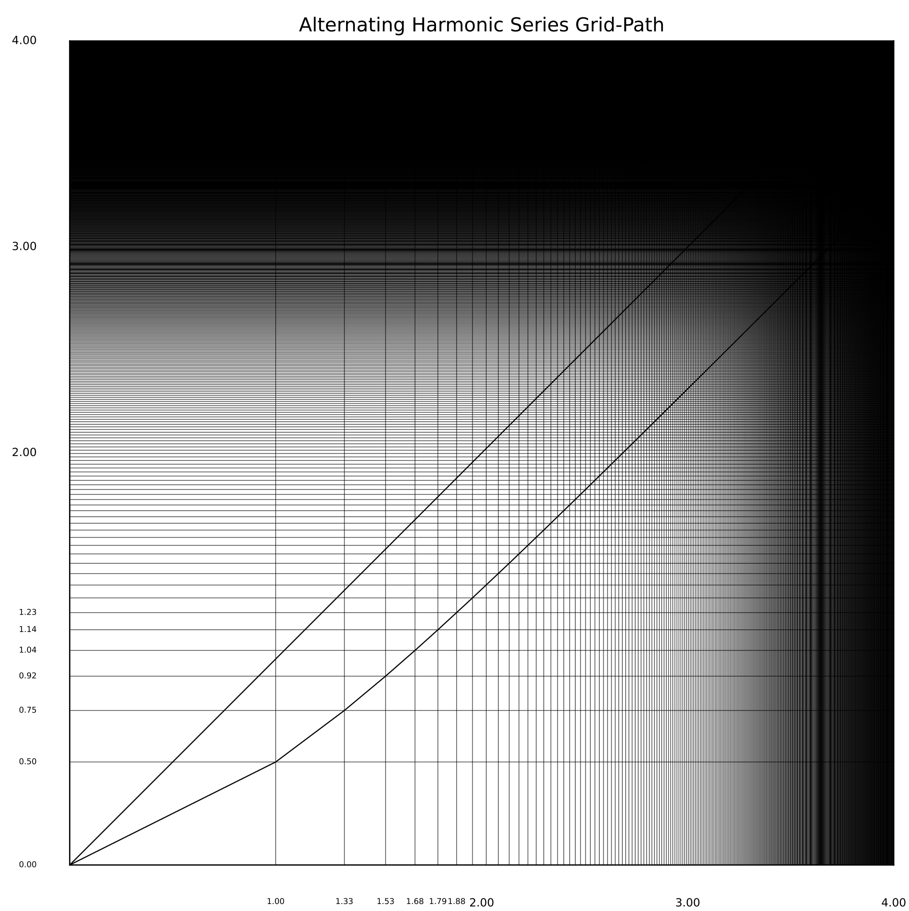


FIGURE 1. Grid Corresponding to $1 - \frac{1}{2} + \frac{1}{3} - \cdots$

identity under composition $u(r) = r$. Our goal is to develop the continuous analogy of a series expressed as an increasing path along the vertices of a lattice in D^2. One can extend G_1, G_2 from before to D by interpolation, with a great degree of freedom. To demonstrate this freedom, again consider a positive (or negative, resp.) partial sum α_n^+. Visualize this sequence as a step function, and our sum as the area below it. Consider all the integrable functions $g : D \to D$ which always satisfy

$$\int_{n-1}^{n} g(t)dt = \alpha_n^+.$$

Every function satisfying this condition will represent α^+ up to $\mathbb{N}$. A pair (g_1, g_2) will represent (α^+, α^-) up to $\mathbb{N}^2$. Their component-wise integral, denoted (G_1, G_2), will represent $(\sum \alpha^+, \sum \alpha^-)$ up to $\mathbb{N}^2$. This motivates embedding D^2 into the first quadrant Q of $\mathbb{C}$ by $(t, s) \mapsto t+is$ with the complex (Euclidean) topology.

$$Q = \{z \in \mathbb{C} \mid \Re(z), \Im(z) \geq 0\}$$

This induces the map $(g_1, g_2) \mapsto g$ where $g : Q \to \mathbb{C}$ by

$$g(z) = g_1(\Re(z)) + ig_2(\Im(z)).$$

Now interpolate the lattice path μ from before. Let $\mu : D \to \mathbb{C}$ by

$$\mu(r) = \mu_1(r) + i\mu_2(r).$$

satisfying $\mu_1, \mu_2 \in \mathbf{A}_0$ (both of their respective components are continuous and strictly increasing from 0 to ∞) and

$$\begin{aligned}\sum_{i=1}^{n} \alpha_n &= \int_0^{\mu_1(n)} g_1(t)dt - \int_0^{\mu_2(n)} g_2(t)dt \\ &= G_1\mu_1(n) - G_2\mu_2(n)\end{aligned}$$

again taking the shorthand $G_j\mu_j(r) = G_j(\mu_j(r))$. This defines the interpolation $W : D \to \mathbb{R}$ of the original series

$$\begin{aligned}W(r) &= G_1\mu_1(r) - G_2\mu_2(r), \\ W'(r) &= g_1\mu_1(r)\mu_1'(r) - g_1\mu_2(r)\mu_2'(r).\end{aligned}$$

Now consider the integrand of the contour integral of g over $\mu[0, r)$.

$$\begin{aligned}g\mu \cdot \mu' &= (g_1\mu_1 + ig_2\mu_2) \cdot (\mu_1' + i\mu_2') \\ &= g_1\mu_1 \cdot \mu_1' - g_2\mu_2 \cdot \mu_2' + i\left[g_1\mu_1 \cdot \mu_2' + g_2\mu_2 \cdot \mu_1'\right]\end{aligned}$$

Notice that the real part of this integrand is exactly $W'(r)$, so

$$\begin{aligned} W(r) &= \int_0^r \Re\left(g\mu(t)\cdot\mu'(t)\right)dt \\ &= \Re\int_0^r g\mu(t)\cdot\mu'(t)dt \\ &= \Re\int_{\mu[0,r)} g(z)dz. \end{aligned}$$

This yields the identity

$$\sum_{k=1}^{n}\alpha_k = \Re\int_{\mu[0,n)} g(z)dz.$$

Example 3.1. Let $\alpha_n = \frac{(-1)^{n-1}}{n}$. Then, using the trigamma function ψ_1, one can extract the corresponding integrands

$$f_1(r) = \frac{1}{2}\psi_1\left(r+\frac{1}{2}\right),\ f_2(r) = 2\psi_1(2r+1) - \frac{1}{2}\psi_1\left(r+\frac{1}{2}\right)$$

The path will be close enough to $\mu(r) = \left(\frac{r}{2},\frac{r}{2}\right)$, so use the latter. If one is worried about this, the correspondence is exact on the even naturals, so the error will be $O\left(\frac{1}{r}\right)$ on D as $r\to\infty$.

$$\begin{aligned} W &= \Re\int_0^r\left(\frac{1}{2}\psi_1\left(\frac{t+1}{2}\right) + 2i\psi_1(t+1) - \frac{i}{2}\psi_1\left(\frac{t+1}{2}\right)\right)\left(\frac{1}{2}+\frac{i}{2}\right)dt \\ &= \Re\Bigg[\left(\frac{1}{2}+\frac{i}{2}\right)\Bigg(\int_{\frac{1}{2}}^{\frac{r+1}{2}}\psi_1(t)\,dt \\ &\qquad + 2i\int_1^{r+1}\psi_1(t)dt - i\int_{\frac{1}{2}}^{\frac{r+1}{2}}\psi_1(t)\,dt\Bigg)\Bigg] \\ &= \frac{1}{2}\int_{\frac{1}{2}}^{\frac{r+1}{2}}\psi_1(t)dt - \int_0^{r+1} + \frac{1}{2}\int_{\frac{1}{2}}^{\frac{r+1}{2}}\psi_1(t)dt \\ &= \int_{\frac{1}{2}}^{\frac{r+1}{2}}\psi_1(t)dt - \int_0^{r+1}\psi_1(t)dt \\ &= \psi_0\left(\frac{r+1}{2}\right) - \psi_0(r+1) + 2\log(2) \\ &= \log(2) + O\left(\frac{1}{r}\right) \\ &\to \log(2). \end{aligned}$$

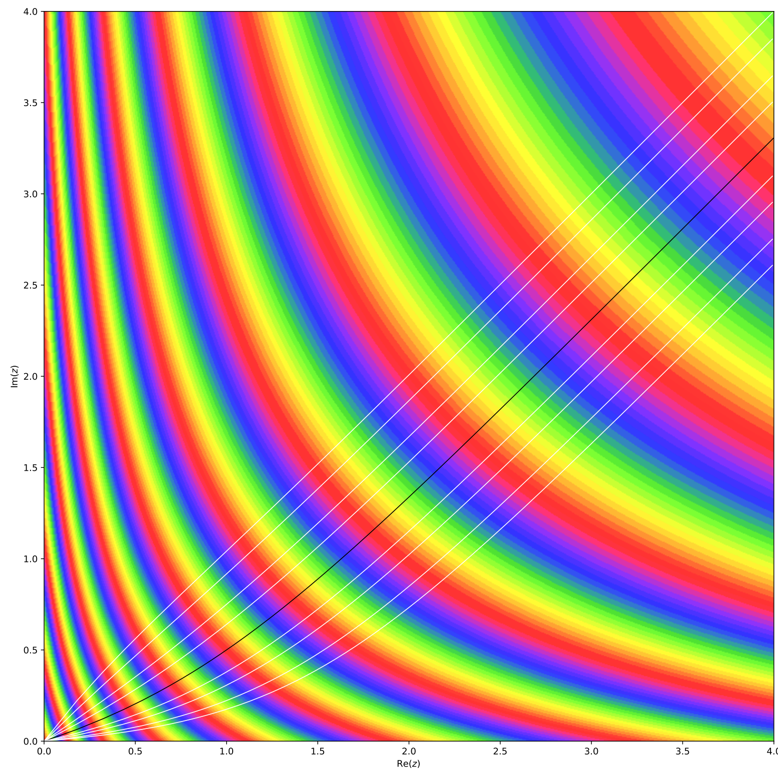


FIGURE 2. Alternating Harmonic Grid-Path Colored by Parameter Level Curves

Moreover, one obtains the following closed form for the total integral:

$$\int_\mu g(z)dz = \psi_0\left(\frac{r+1}{2}\right) - \psi_0(r+1) + 2\log(2) + i(\psi_0(r+1)+\gamma)$$
$$= \sum_{j=1}^{n} \alpha_j + i\sum_{j=1}^{n} |\alpha_j|.$$

With this demonstration concluded, let us define the general case.

3.1. **Axes and Pre-Axes.** Let $\mathbf{A}$ be the set of homeomorphisms $F \in \mathbf{A}_0$ which correspond to one of the signed partial sums of some conditionally convergent series. This occurs precisely when $\int_{n-1}^{n} f(t)dt \to 0$ as $n \to \infty$, but it will be convenient to take the stronger constraint $f(r) \to 0$ a.e. as $r \to \infty$. Because F is increasing and continuous, F will be differentiable (with unique derivative f) except on a set of isolated points. To rid ourselves of this almost-everywhere caveat, define the a.e. representative of the derivative to be 0 whenever F is not differentiable. This gives us a bijection between set $\mathbf{A}$ of asymptotically dense axes,

$$\mathbf{A} = \{F \in \mathbf{A}_0 \mid f(r) \to 0 \text{ as } r \to \infty\}$$

and the set A of their derivatives (called pre-axes)

$$A = \{f : D \to D \mid f(r) \to 0 \text{ as } r \to \infty\}.$$

3.2. **Fields and Grids.** Given two axes, $F_1, F_2 \in \mathbf{A}$, consider the complex function $F : Q \to Q$ which corresponds to the grid generated by the axes F_1, F_2. We accomplish this by demanding that F commutes with $\Re$ and $\Im$, which yields $F(z) = F_1\Re + iF_2\Im$. As such, these F will be called grids. The set $\mathbf{R}_0$ of grids and $\mathbf{R}$ of asymptotically dense grids will be defined

$$\mathbf{R}_0 = \{F : Q \to Q \mid F = F_1\Re + iF_2\Im\colon\ F_1, F_2 \in \mathbf{A}_0\}$$

$$\mathbf{R} = \{F \in \mathbf{R}_0 \mid F_1, F_2 \in \mathbf{A}\}$$

and the set R_0 of fields and R of asymptotically dense fields

$$R_0 = \{f : Q \to Q \mid f = f_1\Re + if_2\Im\colon\ f_1, f_2 \in A_0\}$$

$$R = \{f \in R_0 \mid F_1, F_2 \in A\}.$$

As promised, these are analogous to signed sub-series. That said, a field uniquely (up to a.e. equivalence) defines a grid because the constant of integration is determined by $F(0) = 0$.

3.3. **Paths.** Finally, the set P_0 of paths and P of normal paths by

$$P_0 = \{\gamma : D \to Q \mid \gamma_1, \gamma_2 \in \mathbf{A}_0\}.$$

$$P = \{\gamma \in P_0 \mid \gamma_1 + \gamma_2 = u\}.$$

A normal path is the continuous analogy of a shuffle of a given pair of series. This is why normal paths are parametrized along the taxicab $t + s = r$ instead of the Euclidean $t^2 + s^2 = r^2$. However, a shuffle is more naturally interpolated as a taxicab path along the grid $\mathbb{N}^2$. These are nicely characterized as

$$\mathcal{P} = \big\{\gamma : D \to Q \mid \gamma'(r) \in \{1, i\} \text{ constant for } r \in [n-1, n)\big\}$$

However, they are not pleasant to work with and we will justify using a quotient of P to represent shuffles instead of $\mathcal{P}$.

4. Grid-Paths

Definition 4.1. A pre-field-path is a pair $(f,\gamma) \in R \times P_0$. The real part of the contour integral $\int_\gamma f(z)dz$ is usefully expressed as the work done by the Polya vector field $z \mapsto \overline{f(z)}$ over the path γ, and the imaginary part is the flux of this field over γ.

$$W_\gamma\left[\overline{f}\right](r) = \Re \int_{\gamma[0,r)} f(z)dz$$
$$F_\gamma\left[\overline{f}\right](r) = \Im \int_{\gamma[0,r)} f(z)dz$$

Define the equivalence relation on $(R \times P_0)^2$ by reparametrization:

$$(f,\gamma) \simeq (g,\mu) \text{ iff there exists some } \tau \in \mathbf{A}_0 \text{ s.t. } \gamma(r) = \mu\tau(r)$$

The corresponding classes $\mathrm{gp}(g,\mu) \in (R \times P_0)/\simeq$ will be called grid-paths (or equivalently, field-paths) and the following shorthand will be given

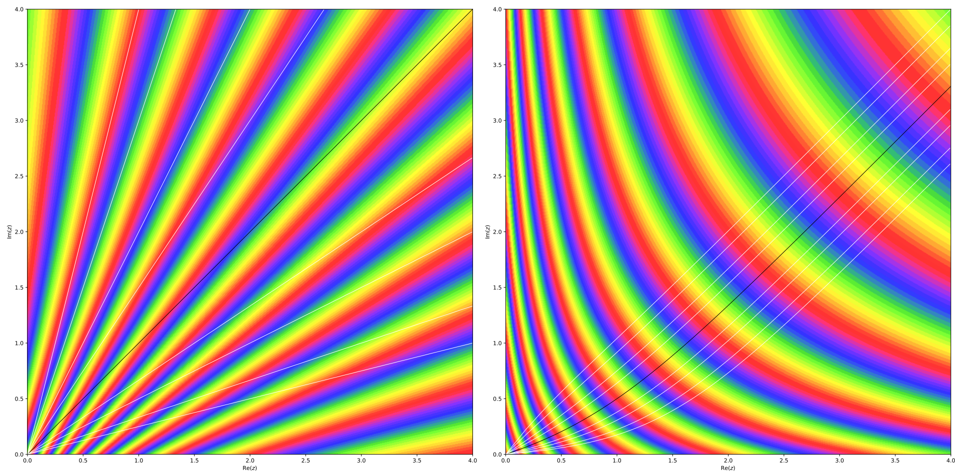
$$\mathbf{GP} = (R \times P_0)/\simeq .$$

This quotient takes on the form

$$\mathbf{GP} \cong R \times P,$$

after choosing the representative $\mathrm{gp}(g,\mu) \in P$ in every class $\mathbf{gp}(g,\mu) \in \mathbf{GP}$.

The equivalence between grid-paths and field-paths is a crucial image to keep in mind. A grid-path $\mathbf{gp}(g,\mu)$ can primarily be visualized as $G\mu \in P_0$, the action on μ by G, where the levels curves are visualized to show how distances are measured. If the grid lines are erased (the parametrization is forgotten) then this looks like any other path in P. On the other hand, it can also be visualized as the vector field $\left[\overline{g}\right]$ acting along $\mu \in P$. These two perspectives will be deemed exterior and interior, respectively, and visualized in figure 3.



Interior Exterior

FIGURE 3. Dual Manifestations of Alternating Harmonic Grid-Path

The following theorem justifies the claim that shuffles can be approximated by normal paths.

Proposition 4.2. *Let* $\alpha \in \mathbf{CC}$. *If* $\iota(\alpha) = \mathbf{gps}(g, \mu)$, *then*

$$\lim_{r \to \infty} \left(W_\mu \left[\overline{g} \right] (r) - \sum_{k=1}^{\lceil r \rceil} \alpha_k \right) = 0.$$

Proof. Consider the set $X \subset D$ of extrema points of $W_\mu \left[\overline{g} \right]$,

$$X = \{ r \in D \mid W_\mu \left[\overline{g} \right]'(r) = 0 \} = \{ x_j \mid j \in \mathbb{N} \},$$

the ordered indexing. Notice that the sequence

$$W_\mu \left[\overline{g} \right] (x_j) - W_\mu \left[\overline{g} \right] (x_{j-1}) \to 0,$$

Suppose this is not the case, then there exists a sequence of positive (or negative) chunks

$$\sum_{k=m_j}^{n_j} \alpha^+ \geq \epsilon > 0,$$

where a chunk is a run of positive (or negative) terms being consecutively added to a partial series. It is easy to see that if the sequence of chunks of a series does not converge to 0, then the series diverges. Thus the series $\sum_{k=1}^{n} \alpha_k$ diverges, a contradiction.

Therefore,

$$W_\mu \left[\overline{g} \right] (x_j) - W_\mu \left[\overline{g} \right] (x_{j-1}) \to 0,$$

and it follows that the error

$$\left|W_\mu\left[\overline{g}\right]-\sum_{k=1}^{\lceil r\rceil}\alpha_k\right|\to 0,$$

as it is always bounded above by the prior quantity for some j where $j\to\infty$ as $r\to\infty$. QED

Definition 4.3. Let $(f,\gamma)\in\mathbf{GP}$. The optimal shuffle $\check{\gamma}_f$ of γ with respect to f is the unique shuffle which minimizes

$$|\hat{\gamma}_f-\gamma| \text{ and } |F\hat{\gamma}_f-F\gamma|,$$

with priority given to the latter. The lattice representative of f is $\check{f}(n)=\int_{n-1}^{n}f(t)dt$ and the set of such lattices

$$\check{R}=\{f:D\to Q\mid f(r)\text{ constant for }r\in[n-1,n)\}.$$

Definition 4.4. Furthering the model, grid-paths are analogous to series. To obtain series exactly, consider the relation $\square$ on $\mathbf{GP}^2$ which equates paths up to their optimal shuffle and lattice integrals

$$\mathrm{gp}(f,\gamma)\ \square\ \mathrm{gp}(g,\mu)\text{ iff }\check{f}=\check{g}\text{ and }\check{\gamma}=\check{\mu}$$

The corresponding classes $\mathbf{gps}(g,\mu)$ will be called grid-path series

$$\mathbf{GPS}=\mathbf{GP}/\square$$

and the representative $\mathrm{gps}(g,\mu)=(\check{g},\check{\mu})$.

Because series can be rearranged to be divergent, a larger set must be defined to contain all these divergent rearrangements in addition to **CC**.

Definition 4.5. Let **CV** denote the set of sequences which tend to 0 and whose absolute series tend to ∞.

$$\mathbf{CV}=\left\{\alpha:\mathbb{N}\to\mathbb{R}\middle|\alpha_n\to 0,\sum_{k=1}^{\infty}|\alpha_n|=\infty\right\}$$

Lemma 4.6.

$$\mathbf{CV}=\mathbf{CC}S_{\mathbb{N}}=\{\alpha\sigma\mid\alpha\in\mathbf{CC},\ \sigma\in S_{\mathbb{N}}\}$$

Proof. For the forward direction, a series $\alpha\in\mathbf{CC}$ has signed sub-series tending to ∞ and summands tending to 0. Therefore, any rearrangement of α must also have these properties. For the backward direction, let $\alpha\in\mathbf{CV}$. Construct σ iteratively to minimize

$$\left|\sum_{k=1}^{n}\alpha_{\sigma_k}\right|$$

for each n. By construction, this quantity can only be as big as the largest summand not yet achieved, so converges (to 0). For a more rigorous treatment of this idea, see the proof of lemma 5.12. QED

Proposition 4.7. $\mathbf{CV} \cong \mathbf{GPS}$ *under the bijection* ι.

Proof. As demonstrated in the prologue, a series $\alpha \in \mathbf{CC}$ can be uniquely decomposed into a pair of signed subsequences and a shuffle between them, and this shuffle is uniquely represented by its linear representative. The same is true of $\alpha \in \mathbf{CV}$, as the assumption of convergence was never used in the construction of the representative. QED

This decomposition naturally embeds $\mathbf{CC}$ into $\mathbf{GPS}$ by the demonstration in the prologue.

Definition 4.8. Define the canonical mapping $\iota : \mathbf{CV} \to \mathbf{GPS}$ by

$$\iota(\alpha) = \mathbf{gps}(g, \mu),$$

where $g_1(r) = \alpha^+_{\lceil r \rceil}$, $g_2(r) = \alpha^-_{\lceil r \rceil}$, $\mu(r)$ the linear representative of the shuffle $\check{\mu_g}$.

Definition 4.9. Let $\alpha \in \mathbf{CV}$. Define its signed multisets by the multiplicity functions $m^\pm : D \to \mathbb{N}$ by

$$m^\pm(r) = \left|\{n \in \mathbb{N} \mid \alpha^\pm_n = r\}\right|.$$

In our view of rearrangements, a series α is a rearrangement of another β if and only if the underlying multisets of the sequences $(\alpha^\pm_n), (\beta^\pm_n)$ are identical (where the signs match accordingly). Notice that multisets are required, as two sequences could have identical underlying sets, but also differing rearrangement orbits due to differing multiplicities.

Let $g \in R$. Define its maximal representative $\hat{g} : \mathbb{N} \to (0, \infty)$ by

$$\hat{G}_j(n) = \max\left\{ \sum_{k \in E} \int_{k-1}^{k} g_j(t)dt \,\middle|\, E \subset \mathbb{N}, |E| = n \right\},$$

which will always be the sum of the n largest elements of the underlying multiset, and thus will be the unique (up to multiset degeneracy) monotone rearrangement of the discrete field $\check{g}$. Consider the relation $\#$ on $\mathbf{GPS}^2$ by

$$\mathbf{gps}(f, \gamma) \ \# \ \mathbf{gps}(g, \mu) \text{ iff } \hat{f} = \hat{g}.$$

Proposition 4.10. *A series* $\alpha = \mathbf{gps}(g,\mu)$ *can be rearranged into a series* $\beta = \mathbf{gps}(f,\gamma)$ *if and only if their maximal (monotone) grids coincide,*

$$\mathbf{mg}(g,\mu) = \mathbf{mg}(f,\gamma).$$

As a consequence, $\mathbf{mg} : \mathbf{GPS} \to \mathbf{MG}$ *maps each series to the set of its rearrangements,*

$$\mathbf{GPS} \cong \mathbf{MG} \times S_{\mathbb{N}}.$$

Proof. QED

From this construction, to find the fixors of a given $\alpha \in \mathbf{CC}$, one may restrict their view to grid-paths which share a maximal grid with α. This also motivates a permutation retrieval strategy, where first each $\mathbf{gps}(f,\gamma) \in \mathbf{GPS}$ is given its canonical signed rearrangement and shuffle with respect to $(\hat{f}, \zeta^{\hat{f}})$, where $\zeta^f \in \mathcal{P}$ is the shuffle which minimizes $|F_1\zeta_1^f(r) - F_2\zeta_2^f(r)|$ and $|\zeta_1^f(r) - \zeta_2^f(r)|$ with priority given to the former. Then each of these permutations composed to the left of σ^{-1}, where $\sigma \in S_{\mathbb{N}}$ is the representative permutation of α in its maximal grid class. Therefore, any given series $\alpha \in \mathbb{C}$ assigns a unique permutation (up to multiset degeneracy) to every series in $\mathbf{CV}$ and any permutation $\sigma \in S_{\mathbb{N}}$ uniquely defines a set of representatives of elements of $\mathbf{MG}$. (each being the σ rearrangement of their $\mathbf{mg}$-rep.) In other words, permutations are a special type of class formed by a choice function over $\mathbf{MG}$ which preserve proximity to the maximal representative.

5. Fixors

Definition 5.1. Let $\alpha \in \mathbf{CC}$. The set $\alpha^\times$ of fixors of α are traditionally given by

$$\alpha^\times = \left\{ \sigma \in S_{\mathbb{N}} \middle| \sum_{n=1}^{\infty} \alpha_{\sigma_n} = \sum_{n=1}^{\infty} \alpha_n \right\}.$$

In our view, if $\iota(\alpha) = \mathbf{gps}(g,\mu) \in \mathbf{GPS}$, the fixors will be given by the set of $\mathbf{gps}$ which share a maximal grid and total work with (g,μ).

$$(g,\mu)^\times = \left\{ \mathbf{gps}(f,\gamma) \in \mathbf{mg}(g,\mu) \mid \mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_\mu\left[\overline{g}\right] \right\}.$$

This is useful, as $\mathcal{W}_\gamma[\overline{\cdot}]$ a well defined linear functional the the set of fields f such that $\mathbf{gps}(f,\gamma) \in \iota(\mathbf{CC})$.

Proposition 5.2. *Let* $\alpha \in \mathbf{CC}$, $\mathbf{gps}(g,\mu) = \iota(\alpha)$.

$$(g,\mu)^\times = \bigsqcup_{\hat{f}=\hat{g}} \left\{ \mathbf{gps}(f,\gamma) \mid \gamma \in P\colon \mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_\mu\left[\overline{g}\right] \right\}$$

though the more concise expression is given by absorbing the grid component into the index:

$$(g,\mu)^\times \cong \coprod_{\hat{f}=\hat{g}} \left\{\gamma \in \mathcal{P} \mid \mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_\mu\left[\overline{g}\right]\right\}.$$

Proof. This result is just a quick corollary of prop. 4.10. By this proposition, the fixors (all permutations in general) of (g,μ) can be decomposed over its monotone grid class, so

$$\begin{aligned}(g,\mu)^\times &= \bigsqcup_{\hat{f}=\hat{g}} \left\{(f,\gamma) \in \mathbf{gps}(g,\mu) \mid \mathcal{W}_\gamma\left[\overline{\{}\right] = \mathcal{W}_\mu\left[\overline{g}\right]\right\}, \\ &\cong \coprod_{\hat{f}=\hat{g}} \left\{\gamma \in \mathcal{P} \mid \mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_\mu\left[\overline{g}\right]\right\}\end{aligned}$$

QED

Definition 5.3. Let $\mathbf{gps}(g,\mu) \in \mathbf{GPS}$, $f \in R$. Define the fixing shuffle class $(g,\mu)^\times_f = \left\{\gamma \in \mathcal{P} \mid \mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_\mu\left[\overline{g}\right]\right\}$, so that the fixors take on the form

$$(g,\mu)^\times = \coprod_{\hat{f}=\hat{g}} (g,\mu)^\times_f.$$

The following theorem is the most crucial for discerning the general geometry of grid-paths.

Theorem 5.4 (Local Work Principle)**.** *Let $(h,\gamma),(g,\mu) \in \mathbf{GP}$. If there exist $\delta_j : D \to \mathbb{R}$ for $j = 1,2,3$ such that*

$$\begin{aligned} &(i)\ H_j(r) = G_j(r + \delta_j(r)) \text{ for } j = 1,2, \\ &(ii)\ \gamma_j(r) = \mu_j(r) + (-1)^j \delta_3(r), \\ &(iii)\ \limsup_{r\to\infty} |\delta_j(r)| < \infty \text{ for } j = 1,2,3, \end{aligned}$$

then $H\gamma(r) - G\mu(r) \to 0$.

Proof. If $H_j(r) = G_j(r + \delta_j(r))$, where $\limsup_{r\to\infty} \delta_j(r) < \infty$, then for any $\gamma \in P$,

$$H_j\gamma_j(r) = G_j(\gamma_j(r) + \delta_j\gamma_j(r)),$$

Let us quickly show that $G_j(\gamma_j(r) + \delta_j\gamma_j(r)) - G_j\gamma_j(r) \to 0$. Clearly $\delta_j\gamma_j$ is still bounded, and $g_j \to 0$, so

$$\begin{aligned} G_j(\gamma_j(r) + \delta_j\gamma_j(r)) - G_j\gamma_j(r) &= \int_{\gamma_j(r)}^{\gamma_j(r)+\delta_j\gamma_j(r)} g_j(t)dt \\ &= O(g_j(t)) \to 0. \end{aligned}$$

Thus from here, it suffices to show that for general γ_j, δ_j from before, $G_j\gamma_j(r) - G_j(\gamma_j(r) + \delta_j(r)) \to 0$, but this is exactly the same condition as before, because evaluating the grid along an outer-deviated path is equivalent to evaluating some inner-deviated grid along the path. QED

Definition 5.5. Let $g \in R$. Let $\delta : D \to Q$. g is said to be $\delta-$tame iff

$$\lim_{r\to\infty} \inf_{\delta\in[0,\delta(r)]} \frac{f(r+\delta)}{f(r)} = \lim_{r\to\infty} \sup_{\delta\in[0,\delta(r)]} \frac{f(r+\delta)}{f(r)} = 1,$$

where the convention $[a, b] \mapsto [b, a]$ whenever $a > b$, so $[0, \delta(r)] \mapsto [\delta(r), 0]$ when $\delta(r) < 0$. If $\Delta \subseteq Q^D$, then f is $\Delta-$tame iff f is δ-tame for some $\delta \in \Delta$.

Corollary 5.6 (Strong Substitution Principle)**.** *Let $h, g \in R$ such that $H(z) - G(z) = O(g(z))$. If g is $O(1)-$tame, then for all $\mu \in P$, $H\mu(r) - G\mu(r) \to 0$. Thus, for all $\mu \in P$, $\hat{f} = \hat{g}$,*

$$(g, \mu)^\times_f = (h, \mu)^\times_f.$$

Proof. The substitution principle is really a special case of the invariant property. Notice that, for tame g,

$$\int_r^{r+O(1)} g_j(t)dt = O(g_j(r)),$$

so for any $\mu \in P$,

$$H\mu(r) = H\mu(r) + \int_{\mu[r,r+O(1))} g(z)dz.$$

We can then see that $H\mu(r) = G(\mu(r) + O(1))$, so by the invariant property, $H\mu(r) - G\mu(r) \to 0$. QED

The prior theorem states that a field h can be substituted in for g whenever finding the work-equivalent grid-paths of (g, μ). (though this does not mean their underlying fixors will be the same) It will be helpful to make such substitution claims about individually specified paths. The following proposition does so.

Proposition 5.7 (Weak Substitution Principle)**.** *Let $h, g \in R$ such that $H(z) - G(z) = O(g(z))$. For all $\mu \in P$ such that $g\mu$ is $o\left(\frac{1}{g\mu}\right)-$tame, $H\mu(r) - G\mu(r) \to 0$.*

Proof. As before, express the difference integral

$$H_j\mu_j(r) - G_j\mu_j(r) = \int_{\mu_j}^{\mu_j(r)+o\left(\frac{1}{g_j\mu_j(r)}\right)} g_j(t)dt,$$

so that tameness immediately yields

$$\int_{\mu_j(r)}^{\mu_j(r)+o\left(\frac{1}{g_j\mu_j(r)}\right)} g_j(t)dt = o\left(\frac{g_j\mu_j(r)}{g_j\mu_j(r)}\right) \to 0.$$

QED

Now, returning to the fixing shuffle classes $(g,\mu)_f^\times$, reparametrization causes a bit of trouble. As such, finding fixors is two-fold. One first must find the pre-grid paths $(f, F^{-1}(G\mu + o(1))) \in R \times P_0$, and reparametrize them individually so that their paths are all in P.

Proposition 5.8. *Let* $(g,\mu) \in \mathbf{GP}$, $f \in R$. *The fixing shuffle class* $(g,\mu)_f^\times$ *is a path-equivalent to the family of paths* $F^{-1}(G\mu + o(1))$,

$$(g,\mu)_f^\times = F^{-1}(G\mu + o(1))/\simeq,$$

where the quotient is the set of $\simeq -$*class representatives.*

Proof. By the relation $\simeq$, QED

These results enable one to start finding fixors. From this point on, the alternating harmonic series will be used as the main example, as it lies just above the main phase transition threshold.

Example 5.9. As before, $\iota(\varrho) = \mathbf{gps}(g,\mu)$, where

$$\begin{aligned} g_1(r) &= \frac{1}{2}\psi_1\left(r+\frac{1}{2}\right) + \frac{\gamma}{2} + \log(2), \\ g_2(r) &= \psi_1(2r+1) - \frac{1}{2}\psi_1\left(r+\frac{1}{2}\right) + \frac{\gamma}{2} - \log(2), \\ \mu &= e. \end{aligned}$$

Notice that $\log(r) = \psi_0(r) + O(r^{-1})$. By the strong substitution principle, one exchange G with $H \in R$ by

$$H_1(r) = \frac{1}{2}\log(r+1) + \frac{\gamma}{2} + \log(2),\ H_2(r) = \frac{1}{2}\log(r+1) + \frac{\gamma}{2} \text{ e.a.}$$

so $h_1(r) = h_2(r)$ eventually always.

To demonstrate the utility of prop. 5.8, let us find all the fixors of ϱ corresponding to the grids

$$F_1(r) = a\log(r+1),\ F_2(r) = b\log(r+1),$$

where $0 < a, b \leq 1/2$.

$$\begin{aligned} a\log(\gamma_1(r)+1) &= \frac{1}{2}\log(r+1) + \frac{\gamma}{2} + \log(2) + o(1), \\ b\log(\gamma_2(r)+1) &= \frac{1}{2}\log(r+1) + \frac{\gamma}{2} + o(1), \end{aligned}$$

and by 5.8, we may allow $\gamma \in P_0$, so that

$$\gamma_1(r) = 2^{\frac{1}{a}} e^{\frac{\gamma}{2a}} r^{\frac{1}{2a}} + o\left(r^{\frac{1}{2a}}\right), \ \gamma_2(r) = e^{\frac{\gamma}{2b}} r^{\frac{1}{2b}} + o\left(r^{\frac{1}{2b}}\right).$$

Now simply solve for the one which will be smaller. In general, $\gamma_1 < \gamma_2$ iff $a > b$. If $a > b$, then

$$(g,\mu)^\times_f = \left\{ \gamma \in \mathcal{P} \middle| \gamma_1(r) = 2r^{\frac{b}{a}} + o\left(r^{\frac{b}{a}}\right) \right\};$$

otherwise

$$(g,\mu)^\times_f = \left\{ \gamma \in \mathcal{P} \middle| \gamma_2(r) = \frac{r^{\frac{a}{b}}}{2} + o\left(r^{\frac{a}{b}}\right) \right\}.$$

Notice that every such path looks like $\left(2r^{\frac{b}{a}}, r - 2r^{\frac{b}{a}}\right)$ at ∞, as the variance between any two paths is asymptotically smaller than their growth. This ceases to be the case for any $f(z) = o\left(\frac{1}{|z|}\right)$, which is essentially the phase transition from the abstract.

Definition 5.10. Let $\gamma \in P_0$. Define $\theta(\gamma)$, the asymptotic class of γ and $\Theta(\gamma)$, the asymptotic order of γ, by

$$\theta(\gamma) = \big\{ \lambda \in P \mid \lambda_j = \gamma_j + o(\gamma_j) \big\},$$
$$\Theta(\gamma) = \big\{ \lambda \in P \mid \lambda_j = O(\gamma_j),\, \gamma_j = O(\lambda_j) \big\}.$$

Notice that θ, Θ extend to any space constructed out of real-valued functions on D.

Proposition 5.11 (Harmonic Phase Transition)**.** *Let* $(g,\mu) \in \mathbf{GP}$. *If* $\frac{1}{|z|} = O(\hat{g}(z))$, *then for every grid* $f \in R$ *on the same asymptotic order as* $\hat{g}$, $(f \in \Theta(\hat{g}))$ *the fixing shuffle class is contained in its own asymptotic class,*

$$(g,\mu)^{\times}_{f} \subseteq \theta(g,\mu)^{\times}_{f}.$$

If the opposite is true, $\hat{g}(z) = o\left(\frac{1}{|z|}\right)$, *then for every* $f \in R$ *on the same asymptotic order as* $\hat{g}$, *the fixing shuffle class contains its own asymptotic order,*

$$\Theta(g,\mu)^{\times}_{f} \subseteq (g,\mu)^{\times}_{f}.$$

Proof. In the first case, Let $\lambda, \gamma \in (g,\mu)^{\times}_{f}$. Then

$$\int_{\gamma_j(r)}^{\lambda_j(r)} f(t)dt \to 0,$$

so by the assumption $\frac{1}{|z|} = O(\hat{g}(z)) = O(f(z))$,

$$\int_{\gamma_j(r)}^{\lambda_j(r)} \frac{1}{t} dt \to 0,$$

i.e.

$$\frac{\lambda_j(r)}{\gamma_j(r)} \to 1.$$

Finally $\lambda_j(r) = \theta(\gamma_j(r))$ for any pair, so

$$(g,\mu)^{\times}_{f} \subseteq \theta(g,\mu)^{\times}_{f}.$$

For the second case, let $\gamma \in (g,\mu)^{\times}_{f}$ and $\lambda \in \Theta(\gamma)$.

$$\begin{aligned}\int_{\gamma_j(r)}^{\lambda_j(r)} f(t)dt &= o\left(\frac{\lambda_j(r)}{\gamma_j(r)}\right) \\ &= o(\Theta(1)) \to 0.\end{aligned}$$

Therefore, $\Theta(g,\mu)^{\times}_{f} \subseteq (g,\mu)^{\times}_{f}$. QED

As figure 4 is beginning to show, subharmonic fields collapse the entire linear quadrant into a single shuffle class, regardless of monotonicity, so long as the monotone grid is subharmonic. The problem of finding fixors becomes quite different for subharmonic fields F, precisely because they fail the property $\frac{1}{g} = o(G)$ from the proofs of the substitutions principles. In this range of fields, shuffles become far less effective at affecting series. If a path γ is bounded between any two lines of positive slope, then on any subharmonic grid f,

$$\mathcal{W}_\gamma\left[\overline{f}\right] = \mathcal{W}_e\left[\overline{f}\right].$$

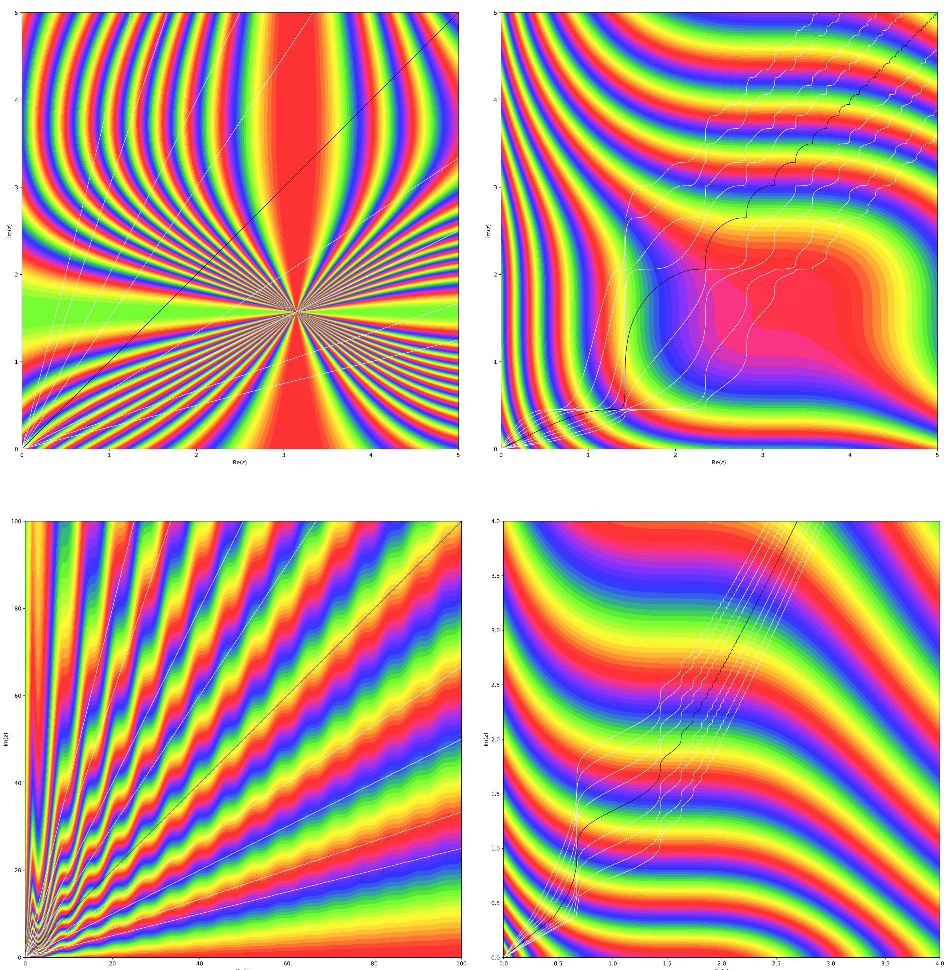


Figure 4. Linear Paths Through Harmonic vs Subharmonic Fields

The keen reader may say something like this happens for every boundary $f = o(g)$ and $g = O(f)$. While this is true, none are so visually apparent, which has to do with the choice of taxicab parametrization when taking a path quotient. Because a harmonic field f has $r = \theta\left(\frac{1}{f_j(r)}\right)$, it is the boundary at which the "radius" of the central fixing class surpasses to the "radius" r of the shuffle space up to r.

Lemma 5.12. *[Maximal Grid Density Lemma] Let $g, \tilde{g} \in R$ such that $\tilde{g} \leq \hat{g}$. There exists some $f \in R$ such that $\hat{f} = \hat{g}$ and $F(r) - \tilde{G}(r) \to 0$.*

Proof. For $\epsilon > 0$, define

$$E(\epsilon) = \{j \in \mathbb{N} \mid \tilde{g}(k) \leq \epsilon\}$$

and the mapping $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(n) = \begin{cases} \min E\left(\frac{\tilde{g}(n)}{2}\right) \setminus \sigma[1, n), & \check{g}(n-1) \leq \sum_{j=1}^{n-1} \tilde{g}(\sigma_j); \\ \min \mathbb{N} \setminus \sigma[1, n), & \tilde{g}(n-1) > \sum_{j=1}^{n-1} \check{g}(\sigma_j). \end{cases}$$

We will demonstrate that this is a bijection by demonstrating that the recursion mapping satisfies the condition of each of its pieces infinitely often. By definition, $\check{f}(0) = \tilde{g}(0)$, so $n = 0$ will always satisfy the first condition. Because $\check{f}(\sigma_n) \leq \frac{\tilde{g}}{2}$,

$$\tilde{g}(n-1) - \sum_{j=1}^{n-1} \check{f}(\sigma_j) \geq \frac{1}{2} \sum_{j=1}^{n-1} \check{f}(j) = \frac{\tilde{g}(n-1)}{2},$$

but the right hand side would diverge if the first recursion iterated arbitrarily, i.e. the second condition will be satisfied after a finite run. And since $\check{F} - \frac{\tilde{G}}{2} \to \infty$, there will be a wealth of early terms to pluck from which will eventually force the rearranged sequence to satisfy the first condition once again. Thus, by induction, each condition is satisfied infinitely often. The mapping is clearly injective, as each $\sigma(n)$ is defined not to be in $\sigma[1, n)$. Furthermore, the second recursion forces surjectivity, as its condition is satisfied infinitely often and it achieves precisely the smallest natural not yet achieved each time. Thus $\sigma \in S_{\mathbb{N}}$, so we need only to prove $F - \tilde{G} \to 0$. We can see that runs satisfying the first condition will yield sums getting monotonically closer to the target $\tilde{G}$, as the summands are defined to be smaller than those of $\tilde{G}$. Runs satisfying the second condition with either monotonically decrease the target gap until it overshoots by less than the largest term not yet summed. Therefore, $F - \tilde{G} \to 0$. QED

Proposition 5.13 (Riemann GPS Theorem)**.** *Let $\alpha \in$ **CC**, $\mathbf{gps}(g, \mu) = \iota(\alpha)$. Let $w : D \to \mathbb{R}$ be integrable such that $w(r) \to 0$ and $W(r) =$*

$\int_0^r w(t)dt$ satisfies

$$-\hat{G}_2\mu_2(r) \leq W(r) \leq \hat{G}_1\mu_1(r).$$

For every $f \in R$ such that $\hat{f} = \hat{g}$, there exists some $\epsilon \in A$ and an uncountable family of rearrangements $\sigma \in S_{\mathbb{N}}$ corresponding to $(f,\gamma) \in$ $\mathbf{mg}(g,\mu)$ which satisfy

$$\sum_{k=1}^{n} \alpha_{\sigma_k} - W(n) = O(\epsilon(n)).$$

This is to say that every limit of vanishing change is obtained by uncountably many rearrangements in every shuffle class of α with nontrivial rate of convergence.

Proof. Let $\hat{f} = \hat{g}$. Let $r \in D$. By the intermediate value theorem, there exists a complex number $t + is$ such that $t + s = r$ and $W(r) = F_1(t) - F_2(s)$. Furthermore, because W and F are continuous, some path $\gamma(r) = s(r) + it(r)$ is as well. Furthermore, let $\lambda \in P$ such that

$$\lambda_j(r) - \gamma_j(r) = o\left(\max_{t\in[\gamma_j(r),\lambda_j(r)]} \frac{1}{f_j(t)}\right).$$

Then

$$\begin{aligned} F_j\lambda_j(r) - F_j\gamma_j(r) &= \int_{\gamma_j(r)}^{\lambda_j(r)} f_j(t)dt \\ &= o(1) \to 0 \end{aligned}$$

Thus, the shuffle class around μ has a "radius" $o\left(\min_{t\in[\gamma_j(r),\lambda_j(r)]} \frac{1}{f_j(t)}\right)$, and since $f_j(r) \to 0$, this radius goes to ∞. Therefore, the construction of an arbitrary permutation in this family requires countable binary choices. (every time this radius increases sufficiently) QED

Theorem 5.14 (Superharmonic Fixing Condition)**.** *Let $f \in R$ $\gamma, \lambda \in$ P. If γ, λ satisfy the condition*

$$\lambda_j(r) - \gamma_j(r) = o\left(\min_{t\in[\gamma_j(r),\lambda_j(r)]} \frac{1}{f_j(t)}\right),$$

then $F\lambda(r) - F\gamma(r) \to 0$. Moreover, if ever $\mathbf{gps}(f,\gamma) \in (g,\mu)_f^{\times}$ or $\mathbf{gps}(f,\gamma) \in (g,\mu)_f^{\times}$, then both inclusions hold.

Proof. As in the prior proof,

$$\begin{aligned} |F_j\lambda_j(r) - F_j\gamma_j(r)| &= \left| \int_{\gamma_j(r)}^{\lambda_j(r)} f_j(t)dt \right| \\ &= o\left(\frac{\max_{s\in[\gamma_j(r),\lambda_j(r)]} f_j(s)}{\max_{t\in[\gamma_j(r),\lambda_j(r)]} f_j(t)} \right) \\ &= o(1) \to 0. \end{aligned}$$

QED

Unfortunately, finding such a condition for subharmonic series is quite challenging beyond $\Theta(g,\mu)^{\times}_f \subseteq (g,\mu)^{\times}_f$. Let us demonstrate some useful formulas for the contour integrals corresponding to grid-paths.

6. Contour Integrals

Proposition 6.1. *Let* $(f,\gamma) \in \mathbf{GP}$. *Then its contour integral is*

$$\int_{\gamma[0,r)} f(z)dz = F_1\gamma_1(r) - F_2\gamma_2(r) + i\left(F_1\gamma_1(r) + F_2\gamma_2(r) + E_\gamma\left[\overline{f}\right](r)\right),$$

where

$$E_\gamma\left[\overline{f}\right](r) = \int_0^r (f_1\gamma_1(t) - f_2\gamma_2(t))(\gamma_2'(t) - \gamma_1'(t))dt.$$

Proof.

$$\begin{aligned} \int_{\gamma[0,r)} f(z)dz &= \int_0^r (f_1\gamma_1(t) + if_2\gamma_2(t))(\gamma_1'(t) + i\gamma_2'(t))dt \\ &= \int_0^r (f_1\gamma_1(t)\gamma_1'(t) - f_2\gamma_2(t)\gamma_2'(t))dt \\ &\quad + i\int_0^r (f_1\gamma_1(t)\gamma_2'(t) + f_2\gamma_2(t)\gamma_1'(t))dt \\ &= F_1\gamma_1(r) - F_2\gamma_2(r) + i(F_1\gamma_1(r) + F_2\gamma_2(r)) \\ &\quad + i\int_0^r (f_1\gamma_1(t) - f_2\gamma_2(t))(\gamma_2'(t) - \gamma_1'(t))dt \\ &= F_1\gamma_1(r) - F_2\gamma_2(r) + i\left(F_1\gamma_1(r) + F_2\gamma_2(r) + iE_\gamma\left[\overline{f}\right](r)\right) \end{aligned}$$

QED

This term $E_\mu\left[\overline{g}\right](r)$ is called the error flux of $\overline{f}$ over γ. In many cases, it turns out, error flux converges to $\mathcal{E}_\mu\left[\overline{g}\right]$. The following proposition provides one sufficient condition for this convergence.

Proposition 6.2. *Let* $(g, \mu) \in \mathbf{GP}$. *If*

$$g_1\mu_1 - g_2\mu_2 \in L^p(D) \text{ and } \mu_1' - \mu_2' \in L^q(D)$$

for Hölder conjugates $p, q \in (1, \infty)$, *then* $\mathcal{E}_\mu\left[\overline{g}\right]$ *converges.*

If $\alpha \in \mathbf{CC}$ *such that* $\iota(\alpha) = \mathbf{gps}(g, \mu)$, *then there exists a unique convergent complex series* $\sum_{k=1}^{\infty} \beta_k$ *such that*

$$\int_{\mu[0,n)} g(z)dz = \sum_{j=1}^{n} \alpha_j + i \sum_{k=1}^{n} |\alpha_k| + \sum_{l=1}^{n} \beta_l \text{ and } \sum_{l=1}^{\infty} \beta_l = i\mathcal{E}_\mu\left[\overline{g}\right].$$

Proof. By Hölder's inequality, $(g_1\mu_1 - g_2\mu_2)(\mu_2' - \mu_1') \in L^1(D)$, so we can immediately see that

$$\mathcal{E}_\mu\left[\overline{g}\right] = \int_0^\infty (g_1\mu_1(t) - g_2\mu_2(t))(\mu_2'(t) - \mu_1'(t))dt \text{ converges.}$$

Let $\iota(\alpha) = \mathbf{gps}(g, \mu)$. By prop. 6.1,

$$\Im \int_{\mu[0,r)} g(z)dz - G_1\mu_1(r) - G_2\mu_2(r) \to \mathcal{E}_\mu\left[\overline{g}\right],$$

and by prop. 4.2,

$$G_1\mu_1(n) + G_2\mu_2(n) - \sum_{k=1}^{n} |\alpha_k| \to 0$$

Now define

$$\beta_n = \int_{\mu[n-1,n)} g(z)dz - \alpha_n - i|\alpha_n|$$

and notice that the prior work shows that

$$\sum_{k=1}^{\infty} \beta_k = i\mathcal{E}_\mu\left[\overline{g}\right],$$

so β is exactly the series we were looking for. QED

This series β is of much interest to grid-paths, as it measures the error between a grid-path and an ideal series

$$\begin{aligned}\sum_{k=1}^{n} \beta_k &= \int_{\mu[0,n)} g(z)dz - \sum_{j=1}^{n} \alpha_j - i \sum_{k=1}^{n} |\alpha_k| \\ &= \int_{\mu[0,n)} g(z)dz - \left[(1+i) \sum_{j=1}^{\check{\mu}_1(n)} \alpha_j^+ + (-1+i) \sum_{k=1}^{\check{\mu}_2(n)} \alpha_k^-\right].\end{aligned}$$

The crux of this model is exploitation of the fact

$$\Re\sum_{k=1}^{\infty}\beta_k = 0,$$

so neglect $\Re\beta_n$ and define its real part, $\alpha^{(g,\mu)} \in \mathbf{CC}$, by

$$\alpha_n^{(g,\mu)} = \Im\int_{\mu[n-1,n)} g(z)dz - i|\alpha_n|.$$

$\alpha^{(g,\mu)}$ will be called the inversion of the α about (g,μ).

Proposition 6.3. *Let $g \in R$. There exists a continuous path $\zeta : D \to Q$ such that $|\zeta(r)| \nearrow \infty$ and for every $r \in D$, G is analytic at $\zeta(r)$. Call the path $\zeta = \zeta_g$ the analytic path of G. If $g \searrow 0$, then ζ_g is unique and is in P.*

Proof. Consider the Cauchy-Riemann Equations on G. Let a general complex variable $z = t + is$.

$$\frac{\partial G_1}{\partial t} = g_1(t),\ \frac{\partial G_1}{\partial s} = 0,$$
$$\frac{\partial G_2}{\partial s} = g_2(s),\ \frac{\partial G_2}{\partial t} = 0,$$

so G is analytic at z if and only if (thanks to the assumption of continuity)

$$g_1(t) = g_2(s).$$

By the intermediate value theorem, there must exist such a solution $t + s = r$ for every $r \in D$, so the analytic path exists. To prove uniqueness in the monotone case, fix $r \in D$ and let $t + s = t' + s' = r$. Suppose $g_1(t) = g_2(s)$ and $g_1(t') = g_2(s')$. Then $g_1(t) - g_1(t') = g_2(s) - g_2(s')$. These must have opposite signs, so $g_1(t) = g_1(t')$ and $g_2(s) = g_2(s')$, so $t = t'$ and $s = s'$. Now label this unique map $r \mapsto t+is$ by $\zeta : D \to Q$. Since $g_1(\zeta_1(r)) = g_2(r - \zeta_1(r))$ for all r, we can simply express $r - \zeta_1(r) = g_2^{-1}g_1\zeta_1(r)$, so either both sides must be decreasing or both increasing. If the former, then $\zeta_1(r) + \zeta_2(r) \neq r$ as the LHS is decreasing, a contradiction. Therefore, both sides are increasing for all r, so $\zeta_j \in \mathbf{A}_0$, i.e. $\zeta \in P$. QED

Definition 6.4. The path $e \in P$ by $e_1(r) = e_2(r) = \frac{r}{2}$ will be called the symmetric path. Correspondingly, every reparametrization of e will be called symmetric.

Proposition 6.5. *Let $(g,\mu) \in \mathbf{GP}$. Then*

$$\mathcal{E}_\mu[\overline{g}] = 0$$

if and only if μ is an analytic path of G almost whenever μ is not parallel to the symmetric path. If ζ_g eventually never intersects with e, then $\mu = \zeta_g$ eventually always or $\mu = e$ eventually always.

Proof. Remember that

$$\mathcal{E}_\mu\left[\overline{g}\right] = \int_0^r (g_1\mu_1(t) - g_2\mu_2(t))(\mu_2'(t) - \mu_1'(t))dt.$$

The target formula holds iff this error integral is 0 a.e., equivalent to $g_1\mu_1(r) = g_2\mu_2(r)$ or $\mu_1'(r) = \mu_2'(r)$ a.e. Thus there could be two distinct paths generated by every intersection of the analytic and symmetric paths. If there are no intersections, then μ must be analytic always or symmetric always. QED

The formula described in prop. 6.5 grants many arithmetical advantages. For example, canonical inversion by the symmetric reflection

$$g^T(z) = i\overline{g}(i\overline{z}) = g_2\Re + ig_1\Im$$

yields the formula

$$W_\mu\left[\overline{g^T}\right] = -W_\mu\left[\overline{g}\right] \text{ and } F_\mu\left[\overline{g^T}\right] = F_\mu\left[\overline{g}\right]$$

when $\mathcal{E}_\mu\left[\overline{g}\right] = 0$. There seems to be a powerful linear structure on fields which emerges when the space is centered on the symmetric path.

7. The Hilbert Space of Fields

Let us extend R to also include homeomorphisms from D to finite increasing graphs in Q.

$$V = \left\{f : D \to Q \mid \exists r_j \in [0, \infty] \text{ s.t. } f_j \in \text{Homeo}(D, [0, r_j))\right\}$$

Equip V with pointwise addition

$$(f + g)(z) = f(z) + g(z)$$

and pointwise scalar multiplication by

$$ae^{i\theta}f = (1 + i)a\frac{f_1 + f_2}{2} + (1 - i)a\cos(\theta)\frac{f_1 - f_2}{2}.$$

Before defining an inner product, consider the functional $\phi : \mathbf{GP} \to \mathbb{C}$

$$\phi_\gamma(f) = \mathcal{W}_\gamma\left[\overline{f}\right] + i\mathcal{E}_\gamma\left[\overline{f}\right].$$

This will be the foundation of the inner product, as it not only measures the deviation of its underlying series, but also the deviation of its path from the symmetric and analytic paths. The inner product will be defined

$$\langle f, g\rangle_\gamma = \phi_\gamma(f)\overline{\phi_\gamma(g)}.$$

It is our hope to develop functional analysis on this space where grids and paths are dual to each other, though this goal is beyond the scope of this paper, so the discussion of Hilbert Spaces will halt here.

8. Conclusion

Originally, this paper was born in service the simple logistical goal: Given a series $\alpha \in \mathbf{CC}$, describe its fixors. Grid-paths grant structure to these fixors, by allowing the decomposition of series/rearrangements into fields/grids and paths. While earlier renditions of this paper saw this geometric model rise from scratch as a model in D^2, the complex geometry adds effortless formality to the correspondence between fields and grids. This correspondence bridges the gap between grid-paths, which document the lengths of absolute series/the differences between partial series, and field-paths, which document the differences between signed partial series/the differences between shuffles. The local work principle grants us two main tools for finding fixors, the substitution princples and the invariant fixing condition. The maximal grid density lemma and Riemann GPS Theorem justify the intuition that series rearrangements are just as dense as grid-paths. (up to the lattice)

The most interesting result is the harmonic phase transition, which grants that every linear shuffle is a fixor of every convergent subharmonic series. In some sense, the problem of finding fixors of subharmonic series becomes solid, whereas the problem is liquid for superharmonic series. The former is far more concerned with the rigid structure of signed rearrangements than the fluid structure of shuffles. We anticipate more progress in the study of signed rearrangements of subharmonic series, which are likely to show that even signed rearrangements are fairly ineffectual in altering series. It is my personal hope to develop a theory of quasi-commutativity of conditionally convergent series, in which addition is defined on grid-paths, instead of grid-path series and rearrangements are defined as the formal classes of grid-paths which they represent.

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